Elliptic Solutions for Higher Order KdV Equations

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Abstract

We study higher order KdV equations from the GL(2,$\mathbb{R}$) $\cong$ SO(2,1) Lie group point of view. We find elliptic solutions of higher order KdV equations up to the ninth order. We argue that the main structure of the trigonometric/hyperbolic/elliptic N-soliton solutions for higher order KdV equations is the same as that of the original KdV equation. Pointing out that the difference is only the time dependence, we find N-soliton solutions of higher order KdV equations can be constructed from those of the original KdV equation by properly replacing the time-dependence. We discuss that there always exist elliptic solutions for all higher order KdV equations.

1 Introduction

The soliton system is taken an interest in for a long time by considering that the soliton equation is the concrete example of the exactly solvable nonlinear differential equation [1–12]. Nonlinear differential equation relates to the interesting non-perturbative phenomena, so that studies of the soliton system are important to unveil mechanisms of various interesting physical phenomena such as those in superstring theories. It is quite surprising that such nonlinear soliton equations can be exactly solvable and have N-soliton solutions. Then we have a dogma that there must be the Lie group structure behind the soliton system, which is a key stone to make nonlinear differential equations exactly solvable.

For the KdV soliton system, the Lie group structure is implicitly built in the Lax operator $L = \partial_x^2 - u(x,t)$. In order to see the Lie group structure, it is appropriate to formulate by using the linear differential operator ∂_x as the Schrödinger representation of the Lie algebra, which naturally comes to use the AKNS formalism [4] for the Lax equation

$$\frac{\partial}{\partial x}\begin{pmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{pmatrix} = \begin{pmatrix} a/2 & -u(x,t) \\ -1 & -a/2 \end{pmatrix}\begin{pmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{pmatrix}.$$

Then the Lie group becomes GL(2,$\mathbb{R}$) $\cong$ SO(2,1) for the KdV equation. An addition formula for elements of this Lie group is the well-known KdV type Bäcklund transformation.

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In our previous papers [13–16], we have studied GL(2,$\mathbb{R}$) $\cong$ SO(2,1) Lie group approach for the unified soliton systems of KdV/mKdV/sinh-Gordon equations. Using the well-know KdV type Bäcklund transformation as the addition formula, we have algebraically constructed N-soliton solutions from various trigonometric/hyperbolic 1-soliton solutions [13,15,16]. Since the Lie group structure of KdV equation is the GL(2,$\mathbb{R}$) $\cong$ SO(2,1), which has elliptic solution, we expect that elliptic N-soliton solutions for the KdV equation can be constructed by using the Bäcklund transformation as the addition formula. We then really have succeeded in constructing elliptic N-soliton solutions [14].

We can interpret this fact in the following way: The KdV equation, which is a typical 2-dimensional soliton equation, has the SO(2,1) Lie group structure and the well-known KdV type Bäcklund transformation can be interpreted as the addition formula of this Lie group. Then the elliptic function appears as a representation of the Bäcklund transformation. While, 2-dimensional Ising model, which is a typical 2-dimensional statistical integrable model, has the SO(3) Lie group structure and the Yang-Baxter relation can be interpreted as the addition formula of this Lie group. Then the elliptic function appears as a representation of the Yang-Baxter relation, which is equivalent to the addition formula of the spherical trigonometry [17, 18]. In 2-dimensional integrable, soliton, and statistical models, there is the SO(2,1)/SO(3) Lie group structure behind the model. As representations of the addition formula, the Bäcklund transformation, and the Yang-Baxter relation, there appears an algebraic function such as the trigonometric/hyperbolic/elliptic functions, which is the key stone to make the 2-dimensional integrable model into the exactly solvable model.

In this paper, we consider Lax type higher order KdV equations and study trigonometric/hyperbolic/elliptic solutions. So far special hyperelliptic solutions for more than the fifth order KdV equation have been vigorously studied by formulating it into the Jacobi's inversion problem [19–24]. Since the Lie group structure GL(2,$\mathbb{R}$) $\cong$ SO(2,1) and the Bäcklund transformation are common even for higher order KdV equations, we expect that there always exist elliptic solutions even for higher order. Then we study to find elliptic solutions up to the ninth order KdV equation, instead of special hyperelliptic solutions. We would like to conclude that we always have elliptic solutions for all higher order KdV equations.

The paper is organized as follows: In section 2, we study trigonometric/hyperbolic solutions for higher order KdV equations. We constract elliptic solutions for higher order KdV equations in section 3. In section 4, we consider the KdV type Bäcklund transformation as an addition formula for solutions of the Weierstrass type elliptic differential equation. In section 5, we study special 1-variable hyperelliptic solutions, and we discuss a relation between such special 1-variable hyperelliptic solutions and our elliptic solutions. We devote a final section to summarize this paper and to give discussions.

2 Trigonometric/hyperbolic solutions for the Lax type higher order KdV equations

Lax pair equations for higher order KdV equations are given by

$$L\psi = \frac{a^2}{4}\psi, \tag{2.1}$$

$$\frac{\partial \psi}{\partial t_{2n+1}} = B_{2n+1}\psi, \tag{2.2}$$

where $L = \partial_x^2 - u$. By using the pseudo-differential operator ∂_x^{-1}, B_{2n+1} are constructed from L in the form [25, 26]

$$B_{2n+1} = \left(\mathcal{L}^{2n+1}\right)_{\geq 0} = \partial_x^{2n+1} - \frac{2n+1}{2} u \partial_x^{2n-1} + \cdots, \tag{2.3}$$

with

$$\mathcal{L} = L^{1/2} = \partial_x - \frac{u}{2}\partial_x^{-1} + \frac{u_x}{4}\partial_x^{-2} + \cdots ,$$

where we denote "≥ 0" to take positive differential operator parts or function parts for general pseudo-differential operators. The integrability condition gives higher order KdV equations

$$\frac{\partial L}{\partial t_{2n+1}} = [B_{2n+1},\, L]. \tag{2.4}$$

As these higher order KdV equations comes from the Lax formalism, these higher order KdV equations are called the Lax type. There are various higher order KdV equations such as the Sawada-Kotera type, which is the higher order generalization of the Hirota form KdV equation [27]. As operators B_{2n+1} are constructed from L, higher order KdV equations also have the same Lie group structure GL(2,$\mathbb{R}$) $\cong$ SO(2,1) as that of the original KdV(=third order KdV) equation. Using $u = z_x$, the KdV type Bäcklund transformation is given in the form

$$z'_x + z_x = -\frac{a^2}{2} + \frac{(z' - z)^2}{2}, \tag{2.5}$$

which comes from Eq.(2.1) only, so that it is valid even for the higher order KdV equations. In the Lie group approach to the soliton system, if we find 1-soliton solutions, we can construct N-soliton solutions from various 1-soliton solutions by the Bäcklund transformation Eq.(2.5) as an addition formula of the Lie group.

For 1-soliton solution of Eq.(2.4), if x and t_{2n+1} come in the combination $X^{(2n+1)} = \alpha x + \beta t_{2n+1}^{\gamma} + \delta$, then if $\gamma \neq 1$, the right-hand side of Eq.(2.4) is a function of only X, while the left-hand side is a function of X and t. Therefore, $\gamma = 1$ is necessary, that is, $X = \alpha x + \beta t_{2n+1} + \delta$. N-soliton solutions are constructed from various 1-soliton solutions by the Bäcklund transformation. Then the main structure of N-soliton solutions, which are expressed with $X_i^{(2n+1)}, (i = 1, 2, \cdots, N)$, takes the same functional forms in higher order KdV equations and in the original KdV equation. The difference is only the time dependence of $X_i = \alpha_i x + \beta_i t_{2n+1} + \delta_i, (i = 1, 2, \cdots, N)$, that is, coefficients β_i. This is valid not only for the trigonometric/hyperbolic N-soliton solutions but also for elliptic N-soliton solutions.

For the trigonometric/hyperbolic N-soliton solutions, we can easily determine the time dependence without knowing details of B_{2n+1}. For dimensional analysis, we have $[\partial_x] = M$, $[u] = M^2$ in the unit of mass dimension M. Further, we notice that $[B_{2n+1},\, L]$ does not contain differential operators but it contains only functions. Then we have

$$\frac{\partial u}{\partial t_{2n+1}} = \partial_x^{2n+1} u + \mathcal{O}(u^2). \tag{2.6}$$

As Eq.(2.6) is the Lie group type differential equation, we take the Lie algebraic limit. Putting $u = \epsilon \hat{u}$ first, Eq.(2.6) takes in the form

$$\epsilon \frac{\partial \hat{u}}{\partial t_{2n+1}} = \epsilon \partial_x^{2n+1} \hat{u} + \mathcal{O}(\epsilon^2 \hat{u}^2), \tag{2.7}$$

and afterwards we take the limit $\epsilon \to 0$, which gives

$$\frac{\partial \hat{u}}{\partial t_{2n+1}} = \partial_x^{2n+1} \hat{u}. \tag{2.8}$$

Then for trigonometric/hyperbolic solutions, we see that x and t_{2n+1} come in a combination $X_i = a_i x + \delta_i \to X_i = a_i x + a_i^{2n+1} t_{2n+1} + \delta_i$ for 1-soliton solutions. In this way, the time-dependence for trigonometric/hyperbolic solutions is easily determined without knowing details of B_{2n+1}. We can then obtain trigonometric/hyperbolic N-soliton solutions for the $(2n+1)$-th order KdV equation from the original KdV N-soliton solutions just by replacing $X_i^{(3)} = a_i x + a_i^3 t_3 + \delta_i \to X_i^{(2n+1)} = a_i x + a_i^{2n+1} t_{2n+1} + \delta_i, (i = 1, 2, \cdots, N)$.

For example, the original third order KdV equation is given by [1]

$$u_{t_3} = u_{3x} - 6uu_x, \tag{2.9}$$

and the fifth order KdV equation is given by [27],

$$u_{t_5} = u_{5x} - 10uu_{3x} - 20u_x u_{2x} + 30u^2 u_x. \tag{2.10}$$

These two equations look quite different, but the 1-soliton solution for the third order KdV equation is given by $z = -a\tanh((ax + a^3 t + \delta)/2)$, while 1-soliton solution for the fifth order KdV equation is given by $z = -a\tanh((ax + a^5 t + \delta)/2)$. In this way, even for any N-soliton solutions, we can obtain the fifth order KdV solution from third order KdV solution just by replacing $X_i^{(3)} = a_i x + a_i^3 t + \delta_i \to X_i^{(5)} = a_i x + a_i^5 t + \delta_i$. See more details in the Wazwaz's nice textbook [27].

However, as we explain in the next section, the way to determine the time dependence by taking the Lie algebraic limit does not applicable for elliptic solutions.

3 Elliptic solutions for the Lax type higher order KdV equations

We consider here elliptic 1-soliton solutions for higher order KdV equations up to ninth order. We first study whether higher order KdV equations reduces to differential equations of the elliptic curves. If a differential equation of the elliptic curve exists, via dimensional analysis, $[\partial_x] = M$, $[u] = M^2$, $[k_3] = M^0$, $[k_2] = M^2$, $[k_1] = M^4$, and $[k_0] = M^6$, that must be the differential equation of the Weierstrass type elliptic curve

$$u_x{}^2 = k_3 u^3 + k_2 u^2 + k_1 u + k_0, \tag{3.1}$$

where $k_i (i = 0, 1, 2, 3)$ are constants. We cannot use the method to take the Lie algebraic limit to find the time dependence of the elliptic 1-soliton solution, because we cannot take $u \to 0$ as $k_0 \neq 0$ is essential in the elliptic case. By differentiating Eq.(3.1), we have the following relations;

$$u_{2x} = \frac{3}{2} k_3 u^2 + k_2 u + \frac{1}{2} k_1, \tag{3.2a}$$

$$u_{3x} = 3k_3 u u_x + k_2 u_x, \tag{3.2b}$$

$$u_{4x} = 3k_3 u u_{2x} + 3k_3 u_x{}^2 + k_2 u_{2x}, \tag{3.2c}$$

$$u_{5x} = 9k_3 u_x u_{2x} + 3k_3 u u_{3x} + k_2 u_{3x}, \tag{3.2d}$$

$$u_{6x} = 12k_3 u_x u_{3x} + 9k_3 u_{2x}{}^2 + 3k_3 u u_{4x} + k_2 u_{4x}, \tag{3.2e}$$

$$u_{7x} = 30k_3 u_{2x} u_{3x} + 15k_3 u_x u_{4x} + 3k_3 u u_{5x} + k_2 u_{5x}, \tag{3.2f}$$

$$u_{8x} = 45k_3 u_{2x} u_{4x} + 30k_3 u_{3x}{}^2 + 18k_3 u_x u_{5x} + 3k_3 u u_{6x} + k_2 u_{6x}. \tag{3.2g}$$

[1]We use the notation $u_x = \partial_x u$, $u_{2x} = \partial_x^2 u$, $\cdots$, throughout the paper.

3.1 Elliptic solution for the third order KdV(original KdV) equation

The third order KdV (original KdV) equation is given by

$$u_{t_3} = u_{3x} - 6uu_x = \left(u_{2x} - 3u^2\right)_x . \tag{3.3}$$

We consider the 1-soliton solution, where x and t come in the combination $X = x + c_3 t_3 + \delta$, then we have

$$u_{2x} - 3u^2 - c_3 u = \frac{k_1}{2}, \tag{3.4}$$

where $k_1/2$ is an integration constant. Further multiplying u_x and integrating, we have the following differential equation of the Weierstrass type elliptic curve

$${u_x}^2 = 2u^3 + k_2 u^2 + k_1 u + k_0, \tag{3.5}$$

where k_2, k_1, and k_0 are constants and c_3 is determined as $c_3 = k_2$, which gives the time-dependence of the 1-soliton solution. If we put $\wp = u/2 + k_2/12$, we have the standard differential equation of the Weierstrass $\wp$ function type

$$\wp_x^2 = 4\wp^3 - g_2\wp - g_3, \tag{3.6}$$

with

$$g_2 = {k_2}^2/12 - k_1/2, \tag{3.7a}$$
$$g_3 = -{k_2}^3/216 + k_1 k_2/24 - k_0/4. \tag{3.7b}$$

Elliptic 1-soliton solution is given by

$$u(x, t_3) = u(X^{(3)}) = 2\wp(X^{(3)}) - \frac{k_2}{6}, \tag{3.8}$$

with

$$X^{(3)} = x + c_3 t_3 + \delta, \quad c_3 = k_2.$$

It should be noted that we must parametrize the differential equation of the Weierstrass type elliptic curve by k_2, k_1, and k_0 instead of g_2 and g_3, because coefficients c_{2n+1} in higher order KdV equations, which determine the time dependence, are expressed with k_2, k_1, and k_0. According to the method of our previous paper, if we find various 1-soliton solutions, we can construct N-soliton solutions [14].

3.2 Elliptic solution for the fifth order KdV equation

The fifth order KdV equation is given by [27],

$$u_{t_5} - \left(u_{4x} - 10uu_{2x} - 5{u_x}^2 + 10u^3\right)_x = 0. \tag{3.9}$$

We consider the elliptic solution, where x and t_5 come in the combination of $X = x + c_5 t_5 + \delta$, which gives

$$c_5 u - \left(u_{4x} - 10uu_{2x} - 5{u_x}^2 + 10u^3\right) + C = 0, \tag{3.10}$$

where C is an integration constant. We will show that the above equation reduces to the same differential equation of the Weierstrass type elliptic curve Eq.(3.1). Substituting Eq.(3.1),

$\cdots$, and Eq.(3.2c) into Eq.(3.10) and comparing coefficients of u^3, u^2, u^1, and u^0, we have 4 conditions for 6 constants k_3, k_2, k_1, k_0, c_5, and C in the form

$$\text{i)} \quad (k_3-2)(3k_3-2)=0, \tag{3.11a}$$
$$\text{ii)} \quad k_2(k_3-2)=0, \tag{3.11b}$$
$$\text{iii)} \quad c_5=(9k_3/2-10)k_1+{k_2}^2, \tag{3.11c}$$
$$\text{iv)} \quad C=(3k_3-5)k_0+k_1k_2/2. \tag{3.11d}$$

Then we have two solutions

$$\text{I)} \quad k_3=2, \quad k_2,k_1,k_0:\text{arbitrary}, \quad c_5=-k_1+{k_2}^2, \quad C=k_0+k_1k_2/2, \tag{3.12}$$
$$\text{II)} \quad k_3=\frac{2}{3}, \quad k_2=0, \quad k_1,k_0:\text{arbitrary}, \quad c_5=-7k_1, \quad C=-3k_0. \tag{3.13}$$

We here take the most general solution, i.e. I) case, which gives the same differential equation of the elliptic curve ${u_x}^2=2u^3+k_2u^2+{k_2}^2u+k_0$ as that of the third order KdV equation Eq.(3.5) and c_5 is determined as $c_5=-k_1+{k_2}^2$. Elliptic 1-soliton solution is given by

$$u(x,t_5)=u(X^{(5)})=2\wp(X^{(5)})-\frac{k_2}{6}, \tag{3.14}$$

with

$$X^{(5)}=x+c_5t_5+\delta, \quad c_5=-k_1+{k_2}^2.$$

3.3 Elliptic solution for the seventh order KdV equation

The seventh order KdV equation is given by [27],

$$u_{t_7}-\left(u_{6x}-14uu_{4x}-28u_xu_{3x}-21{u_{2x}}^2+70u^2u_{2x}+70u{u_x}^2-35u^4\right)_x=0. \tag{3.15}$$

In this case, assuming that x and t_7 come in the combination of $X=x+c_7t_7+\delta$, we have

$$c_7u-\left(u_{6x}-14uu_{4x}-28u_xu_{3x}-21{u_{2x}}^2+70u^2u_{2x}+70u{u_x}^2-35u^4\right)+C=0. \tag{3.16}$$

Repeatedly substituting Eq.(3.1), $\cdots$, and Eq.(3.2e) into Eq.(3.16) and comparing coefficients of u^4, u^3, u^2, u^1, and u^0, we have 5 conditions for 6 constants k_3, k_2, k_1, k_0, c_7, and C of the form

$$\text{i)} \quad (k_3-2)(3k_3-2)(3k_3-1)=0, \tag{3.17a}$$
$$\text{ii)} \quad k_2(k_3-2)(3k_3-2)=0, \tag{3.17b}$$
$$\text{iii)} \quad k_1(k_3-2)(6k_3-5)+3{k_2}^2(k_3-2)=0, \tag{3.17c}$$
$$\text{iv)} \quad c_7=(45{k_3}^2-126k_3+70)k_0+(27k_3-56)k_1k_2+{k_2}^3, \tag{3.17d}$$
$$\text{v)} \quad C=(15k_3-28)k_0k_2+(9k_3-21){k_1}^2/4+k_1{k_2}^2/2. \tag{3.17e}$$

Then we get 3 solutions

$$\text{I)} \quad k_3=2, \quad k_2,k_1,k_0:\text{arbitrary}, \quad c_7=-2k_0-2k_1k_2+{k_2}^3,$$
$$C=2k_0k_2-3{k_1}^2/4+k_1{k_2}^2/2, \tag{3.18}$$
$$\text{II)} \quad k_3=2/3, \quad k_1=3{k_2}^2, \quad k_2,k_0:\text{arbitrary}, \quad c_7=6k_0-113{k_2}^3,$$
$$C=-18k_0k_2-129{k_2}^4/4, \tag{3.19}$$
$$\text{III)} \quad k_3=1/3, \quad k_2=0, \quad k_1=0, \quad k_0:\text{arbitrary}, \quad c_7=33k_0, \quad C=0. \tag{3.20}$$

We take the most general solution i.e. I) case, which is the same differential equation of the elliptic curve as that of the third order KdV equation Eq.(3.5) and c_7 is determined as $c_7 = -2k_0 - 2k_1k_2 + {k_2}^3$. Elliptic 1-soliton solution is given by

$$u(x, t_7) = u(X^{(7)}) = 2\wp(X^{(7)}) - \frac{k_2}{6}, \tag{3.21}$$

with

$$X^{(7)} = x + c_7t_7 + \delta, \quad c_7 = -2k_0 - 2k_1k_2 + {k_2}^3.$$

3.4 Elliptic solution for the ninth order KdV equation

The ninth order KdV equation is given by [28],

$$\begin{aligned} u_{t_9} - \big(&u_{8x} - 18uu_{6x} - 54u_xu_{5x} - 114u_{2x}u_{4x} - 69{u_{3x}}^2 + 126u^2u_{4x} + 504uu_xu_{3x} \\ &+462{u_x}^2u_{2x} + 378u{u_{2x}}^2 - 630u^2{u_x}^2 - 420u^3u_{2x} + 126u^5\big)_x = 0. \end{aligned} \tag{3.22}$$

Assuming that x and t_9 come in the combination of $X = x + c_9t_9 + \delta$, we have

$$\begin{aligned} c_9u - \big(&u_{8x} - 18uu_{6x} - 54u_xu_{5x} - 114u_{2x}u_{4x} - 69{u_{3x}}^2 + 126u^2u_{4x} + 504uu_xu_{3x} \\ &+462{u_x}^2u_{2x} + 378u{u_{2x}}^2 - 630u^2{u_x}^2 - 420u^3u_{2x} + 126u^5\big) + C = 0. \end{aligned} \tag{3.23}$$

Substituting Eq.(3.1), $\cdots$, and Eq.(3.2g) into Eq.(3.23) and comparing coefficients of u^5, u^4, u^3, u^2, u^1, and u^0, we have 6 conditions for 6 constants k_3, k_2, k_1, k_0, c_9, and C in the following form

$$\text{i)} \quad (k_3 - 2)(3k_3 - 2)(3k_3 - 1)(5k_3 - 1) = 0, \tag{3.24a}$$

$$\text{ii)} \quad k_2(k_3 - 2)(3k_3 - 2)(3k_3 - 1) = 0, \tag{3.24b}$$

$$\text{iii)} \quad k_1(k_3 - 2)(3k_3 - 2)(9k_3 - 4) + 7{k_2}^2(k_3 - 2)(3k_3 - 2) = 0, \tag{3.24c}$$

$$\begin{aligned} \text{iv)} \quad &3k_0(k_3 - 2)(225{k_3}^2 - 252k_3 + 70) \\ &+ k_2(k_3 - 2)(720k_1k_3 - 546k_1 + 85{k_2}^2) = 0, \end{aligned} \tag{3.24d}$$

$$\begin{aligned} \text{v)} \quad &c_9 = (675{k_3}^2 - 1836k_3 + 966)k_0k_2 + (378{k_3}^2 - 1080k_3 + 651){k_1}^2/2 \\ &+(243k_3 - 492)k_1{k_2}^2/2 + {k_2}^4, \end{aligned} \tag{3.24e}$$

$$\begin{aligned} \text{vi)} \quad &C = (297{k_3}^2 - 828k_3 + 462)k_0k_1/2 + (63k_3 - 123)k_0{k_2}^2 \\ &+(27k_3 - 57){k_1}^2k_2/2 + k_1{k_2}^3/2. \end{aligned} \tag{3.24f}$$

Then we obtain 4 solutions

$$\begin{aligned} \text{I)} \quad &k_3 = 2, \quad k_2, k_1, k_0 : \text{arbitrary}, \quad c_9 = -6k_0k_2 + 3{k_1}^2/2 - 3k_1{k_2}^2 + {k_2}^4, \\ &C = -3k_0k_1 + 3k_0{k_2}^2 - 3{k_1}^2k_2/2 + k_1{k_2}^3/2, \end{aligned} \tag{3.25}$$

$$\begin{aligned} \text{II)} \quad &k_3 = 2/3, \quad k_0 = (66k_1k_2 - 85{k_2}^3)/6, \quad k_2, k_1 : \text{arbitrary}, \\ &c_9 = (99{k_1}^2 + 594k_1{k_2}^2 - 1188{k_2}^4)/2, \quad C = (423{k_1}^2k_2 - 2376k_1{k_2}^3 + 2295{k_2}^5)/2, \end{aligned} \tag{3.26}$$

$$\begin{aligned} \text{III)} \quad &k_3 = 1/3, \quad k_1 = 7{k_2}^2, \quad k_0 = 187{k_2}^3/3, \quad k_2 : \text{arbitrary}, \quad c_9 = 33462{k_2}^4, \\ &C = 40248{k_2}^5, \end{aligned} \tag{3.27}$$

$$\text{IV)} \quad k_3 = 1/5, \quad k_2 = 0, \quad k_1 = 0, \quad k_0 = 0, \quad c_9 = 0, \quad C = 0. \tag{3.28}$$

We take the most general solution i.e. I) case, which gives the same differential equation of the elliptic curve as that of the third order KdV equation Eq.(3.5), and c_9 is determined as $c_9 = -6k_0k_2 + 3{k_1}^2/2 - 3k_1{k_2}^2 + {k_2}^4$. Elliptic 1-soliton solution is given by

$$u(x, t_9) = u(X^{(9)}) = 2\wp(X^{(9)}) - \frac{k_2}{6}, \tag{3.29}$$

with

$$X^{(9)} = x + c_9 t_9 + \delta, \quad c_9 = -6k_0k_2 + 3{k_1}^2/2 - 3k_1{k_2}^2 + {k_2}^4.$$

In this way, even for higher order KdV equations, the main structure of the elliptic solution, which is expressed by $X^{(2n+1)}$, takes the same functional form except the time dependence, that is, c_{2n+1} in $X^{(2n+1)} = x + c_{2n+1}t_{2n+1} + \delta$. Compared with the trigonometric/hyperbolic case, c_{2n+1} becomes complicated for elliptic solutions of higher order KdV equations.

In the general $(2n+1)$-th order KdV equation, by dimensional analysis $[u_{2nx}] = [u^{n+1}] = M^{2n+2}$, integrated differential equation gives the $(n+1)$-th order polynomial of u. Then the number of the conditions is $n+2$, while the number of constants is 6. So, $n \geq 5$ becomes the overdetermined case, but we expect the existence of the differential equation of the elliptic curve for more than eleventh order KdV equation owing to the nice SO(2,1) Lie group symmetry. Although the existence of such elliptic curve is a priori not guaranteed, we will show later that the elliptic solutions really exist for all higher order KdV equations.

4 Bäcklund transformation for the differential equation of the elliptic curve

Here we will show that the Bäcklund transformation connects one solution to another solution of the same differential equation of the Weierstrass type elliptic curve. The Lie group structure of KdV equation is given by GL(2,$\mathbb{R}$) $\cong$ SO(2,1) and the Bäcklund transformation can be considered as the self gauge transformation of this Lie group. We consider two elliptic solutions for the KdV equation, that is, two solutions $u'(x,t_3)$ and $u(x,t_3)$ for $u'_{t_3} - u'_{xxx} + 6u'u'_x = 0$ and $u_{t_3} - u_{xxx} + 6uu_x = 0$. We put the time dependence in the forms; $X' = x + c'_3 t_3 + \delta'$ for $u'(x,t_3)$ and that of $X = x + c_3t_3 + \delta$ for $u(x,t_3)$. In order to connect two solutions by the Bäcklund transformation and to construct N-soliton solutions, c'_3 and c_3 must take the same common value. By integrating twice, we have the same differential equation of the elliptic curve

$$u'^{\,2}_x = 2u'^3 + k_2u'^2 + k_1u' + k_0, \tag{4.1}$$

$$u_x^{\,2} = 2u^3 + k_2u^2 + k_1u + k_0, \tag{4.2}$$

with same coefficients k_2, k_1, and k_0, where we take $c_3 = c'_3 = k_2$. By taking a constant shift of $u \to u - k_2/6$, we consider the same two differential equations of the Weierstrass type elliptic curve

$$u'^{\,2}_x = 2u'^3 - 2g_2u' - 4g_3, \tag{4.3}$$

$$u_x^{\,2} = 2u^3 - 2g_2u - 4g_3, \tag{4.4}$$

where g_2 and g_3 are given by Eqs.(3.7a) and (3.7b). It should be mentioned that this differential equation of the Weierstrass type elliptic curve has not only the solution $u(x) = 2\wp(x)$ but also N-soliton solutions [14].

Here we will show that we can connect two solutions of Eqs.(4.3) and (4.4) by the following Bäcklund transformation

$$z'_x + z_x = -\frac{a^2}{2} + \frac{(z'-z)^2}{2}, \tag{4.5}$$

where $u = z_x$ and $u' = z'_x$. We introduce $U = u' + u = z'_x + z_x$ and $V = z' - z$, which gives $V_x = z'_x - z_x = u' - u$. Then we have $u' = (U + V_x)/2$ and $u = (U - V_x)/2$. Eqs.(4.3) and (4.4) are given by

$$(U_x + V_{xx})^2 = (U + V_x)^3 - 4g_2(U + V_x) - 16g_3, \tag{4.6}$$

$$(U_x - V_{xx})^2 = (U - V_x)^3 - 4g_2(U - V_x) - 16g_3. \tag{4.7}$$

The Bäcklund transformation (4.5) is given by

$$U = \frac{V^2}{2} - \frac{a^2}{2}, \tag{4.8}$$

which gives $U_x = VV_x$.

First, by taking Eq.(4.6)−Eq.(4.7), we have

$$U_x V_{xx} = \frac{1}{2}\left(3U^2 V_x + {V_x}^3\right) - 2g_2 V_x, \tag{4.9}$$

which reads the form

$$VV_{xx} = \frac{3}{8}\left(V^2 - a^2\right)^2 + \frac{1}{2}{V_x}^2 - 2g_2 = \frac{1}{2}{V_x}^2 + \frac{3}{8}V^4 - \frac{3}{4}a^2V^2 + \frac{3}{8}a^4 - 2g_2, \tag{4.10}$$

throught the relation (4.8). By dimensional analysis, we have

$$V_x^2 = m_4V^4 + m_3V^3 + m_2V^2 + m_1V + m_0, \tag{4.11}$$

where $m_i (i = 0, 1, \cdots, 4)$ are constants. By differentiating this relation, we have

$$V_{xx} = 2m_4V^3 + \frac{3}{2}m_3V^2 + m_2V + \frac{1}{2}m_1. \tag{4.12}$$

Substituting this relation into Eq.(4.10), we have

$$2m_4V^4 + \frac{3}{2}m_3V^3 + m_2V^2 + \frac{1}{2}m_1V = \frac{1}{2}{V_x}^2 + \frac{3}{8}V^4 - \frac{3}{4}a^2V^2 + \frac{3}{8}a^4 - 2g_2, \tag{4.13}$$

which gives

$$\begin{aligned} {V_x}^2 &= \left(4m_4 - \frac{3}{4}\right)V^4 + 3m_3V^3 + \left(2m_2 + \frac{3}{2}a^2\right)V^2 + m_1V - \frac{3}{4}a^4 + 4g_2 \\ &= m_4V^4 + m_3V^3 + m_2V^2 + m_1V + m_0. \end{aligned} \tag{4.14}$$

Comparing coefficients of the power of V, we have $m_4 = 1/4$, $m_3 = 0$, $m_2 = -3a^2/2$, $m_1 =$ (undetermined), $m_0 = -3a^4/4 + 4g_2$, which gives

$${V_x}^2 = \frac{1}{4}V^4 - \frac{3}{2}a^2V^2 + m_1V - \frac{3}{4}a^4 + 4g_2, \tag{4.15}$$

$$V_{xx} = \frac{1}{2}V^3 - \frac{3}{2}a^2V + \frac{1}{2}m_1. \tag{4.16}$$

Second, by taking Eq.(4.6)+Eq.(4.7), we have

$${U_x}^2 + {V_{xx}}^2 = U^3 + 3U{V_x}^2 - 4g_2U - 16g_3. \tag{4.17}$$

Using Eq.(4.8), we have

$$V^2{V_x}^2 + {V_{xx}}^2 = \left(\frac{V^2}{2} - \frac{a^2}{2}\right)^3 + 3\left(\frac{V^2}{2} - \frac{a^2}{2}\right){V_x}^2 - 4g_2\left(\frac{V^2}{2} - \frac{a^2}{2}\right) - 16g_3. \tag{4.18}$$

Substituting ${V_x}^2$ and V_{xx} into Eq.(4.18) and by using Eq.(4.15) and Eq.(4.16), we have the condition ${m_1}^2 = 4a^6 - 16a^2g_2 - 64g_3$. Then the undetermined coefficient m_1 is determined, and we have the differential equation of the Jacobi type elliptic curve for $V = z' - z$

$${V_x}^2 = \frac{1}{4}V^4 - \frac{3}{2}a^2V^2 \pm \sqrt{4a^6 - 16a^2g_2 - 64g_3}\,V - \frac{3}{4}a^4 + 4g_2. \tag{4.19}$$

In this way, the set of equations $\Big\{\text{Eq.}(4.3),\ \text{Eq.}(4.5)\Big\}$ is equivalent to the set of those $\Big\{\text{Eq.}(4.4),\ \text{Eq.}(4.5)\Big\}$. This means that the Bäcklund transformation (4.5) connects one soliton solution u to another soliton solution u' for the same differential equation Eq.(4.3) and Eq.(4.4) of the Weierstrass type elliptic curve. In order to construct N-soliton solutions of the $(2n+1)$-th order KdV equation by the Bäcklund transformation, the time dependence for each 1-soliton solution, $c_{2n+1\,i}\,(i=1,2,\cdots,N)$, must be the same common value, then x and t_{2n+1} come in the combination $X_i^{(2n+1)} = x + c_{2n+1}t_{2n+1} + \delta_i$.

In our previous work [14], by using the explicit soliton solution given by $\wp$-function and ζ-function, we connect one soliton solution to another soliton solution by the Bäcklund transformation. Here we have shown that Bäcklund transformation connects one soliton solution to another soliton solution of the same differential equation of the Weierstrass type elliptic curve without using the explicit expression of the solution.

5 Special hyperelliptic solutions for higher order KdV equations

By using the method of commutative ordinary operators [19, 20], we can formulate higher order KdV equations into the Jacobi's inversion problem. By solving the general Jacobi's inversion problem, we can find solutions for higher order KdV equations [20–24]. Here we consider the fifth order KdV equation in order to explain how to solve the Jacobi's inversion problem. Integrated fifth order KdV equation is given by

$$u_{4x} - 10uu_{2x} - 5{u_x}^2 + 10u^3 = c_5 u + C. \tag{5.1}$$

According to the Tanaka-Date's nice paper [20], this fifth order KdV equation is reformulated in the following form. We introduce auxiliary fields $\mu_1(x), \mu_2(x)$,

$$u(x) = 2(\mu_1(x) + \mu_2(x)), \tag{5.2}$$

$$\mu_1(x)_x = \frac{\pm 2\sqrt{f_5(\mu_1(x))}}{\mu_1(x) - \mu_2(x)}, \tag{5.3}$$

$$\mu_2(x)_x = \frac{\pm 2\sqrt{f_5(\mu_2(x))}}{\mu_2(x) - \mu_1(x)}, \tag{5.4}$$

$$f_5(\lambda) = \lambda^5 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0, \tag{5.5}$$

where α_3, α_2, α_1, and α_0 are constants. Surprisingly, this $u(x)$ satisfies

$$u_{4x} - 10uu_{2x} - 5{u_x}^2 + 10u^3 = -8\alpha_3 u + 16\alpha_2. \tag{5.6}$$

which determines $c_5 = -8\alpha_3, C = 16\alpha_2$. Then if we can find the solution $\mu_1(x), \mu_2(x)$, we can construct the solution $u(x,t)$ of the fifth order KdV equation by $u(x,t) = u(X^{(5)}) = 2\left(\mu_1(X^{(5)}) + \mu_2(X^{(5)})\right)$ where $X^{(5)} = x + c_5 t_5 + \delta$.

Eqs.(5.3) and (5.4) can be written in the form of the genus two Jacobi's inversion problem [29]

$$\frac{\mathrm{d}\mu_1(x)}{\sqrt{f_5(\mu_1(x))}} + \frac{\mathrm{d}\mu_2(x)}{\sqrt{f_5(\mu_2(x))}} = 0, \tag{5.7}$$

$$\frac{\mu_1(x)\,\mathrm{d}\mu_1(x)}{\sqrt{f_5(\mu_1(x))}} + \frac{\mu_2(x)\,\mathrm{d}\mu_2(x)}{\sqrt{f_5(\mu_2(x))}} = \pm 2\,\mathrm{d}x. \tag{5.8}$$

The solution of the Jacobi's inversion problem is that the symmetric combination of $\mu_1(x)$ and $\mu_2(x)$, that is , $\mu_1(x) + \mu_2(x) (= u(x)/2)$ and $\mu_1(x)\mu_2(x)$ are given by the ratio of the genus two hyperelliptic theta function. However, the above Jacobi's inversion problem is special as the right-hand side of Eq.(5.7) is zero. Then the genus two hyperelliptic theta function takes in the following special 1-variable form $\vartheta(\pm 2x + d_1, d_2)$ where d_1, d_2 are constants, that is, the second argument becomes constant. Then the ratio of such special genus two hyperelliptic theta function is the function of 1-variable x, which becomes proportional to the 1-variable function $u(x) = 2(\mu_1(x) + \mu_2(x))$. The general genus two hyperelliptic theta function is given by

$$\vartheta(u, v; \tau_1, \tau_2, \tau_{12}) = \sum_{m,n \in \mathbb{Z}} \exp\Big[i\pi(\tau_1 m^2 + \tau_2 n^2 + 2\tau_{12} mn) + 2i\pi(mu + nv) \Big]. \tag{5.9}$$

Then $F(x, t) = \vartheta(x, d_2; t, \tau_2, \tau_{12})$ satisfies the diffusion equation $\partial_t F(x, t) = -i\partial_x^2 F(x, t)/4\pi$. Further, $F(x, t)$ has the trivial periodicity $F(x + 1, t) = F(x, t)$. It is shown in the Mumford's nice textbook [30] that if $F(x, t)$ satisfies i) periodicity $F(x + 1, t) = F(x, t)$, ii) diffusion equation $\partial_t F(x, t) = -i\partial_x^2 F(x, t)/4\pi$, $F(x, t)$ becomes the genus one elliptic theta function of 1-variable x. By solving the Jacobi's inversion problem, the solution $u(x, t_5) = u(X^{(5)}) = u(x + c_5 t_5 + \delta)$ of the fifth order KdV equation is given by the ratio of the special 1-variable hyperelliptic theta function, which gives the elliptic solution. For the $(2n + 1)$-th order KdV equation, the solution of the Jacobi's inversion problem gives $u(x, t_{2n+1}) = u(X^{(2n+1)})$ as the ratio of the special 1-variable genus n hyperelliptic theta function of the form $\vartheta(\pm 2x + d_1, d_2, \cdots, d_n)$, which also becomes the genus one elliptic theta function.

For higher order KdV equations, it is shown that solutions are expressed with above special 1-variable hyperelliptic theta functions, which becomes elliptic theta functions. Then we can conclude that all higher order KdV equations always have elliptic solutions, though we have explicitly constructed elliptic solutions only up to the ninth order KdV equation.

6 Summary and Discussions

We have studied to construct N-soliton solution for the Lax type higher order KdV equations by using the GL(2,$\mathbb{R}$) $\cong$ SO(2,1) Lie group structure. The main structure of N-soliton solutions, expressed with $X_i = \alpha_i x + \beta_i t + \delta_i, (i = 1, 2, \cdots, N)$ is the same even for higher order KdV equations. The difference of N-soliton solutions in various higher order KdV equations is the time dependence, that is, coefficients β_i.

In trigonometric/hyperbolic solutions, by taking the Lie algebra limit, we can easily determine the time dependence. For the $(2n + 1)$-th order KdV equation, we can obtain N-soliton solutions from those of the original KdV equation by just the replacement $X_i^{(3)} = a_i x + a_i^3 t_3 + \delta_i$ $\to X_i^{(2n+1)} = a_i x + a_i^{2n+1} t_{2n+1} + \delta_i$, $(i = 1, 2, \cdots, N)$.

For elliptic solutions, up to the ninth order KdV equation, we have obtained N-soliton solutions from those of the original KdV equation by just the replacement ${X^{(3)}}_i = x + c_3 t_3 + \delta_i$ $\to {X^{(2n+1)}}_i = x + c_{2n+1} t_{2n+1} + \delta_i$, $(i = 1, 2, 3, 4)$ where c_{2n+1} are given by $c_3 = k_2$, $c_5 = -k_1 + {k_2}^2$, $c_7 = -2k_0 - 2k_1 k_2 + {k_2}^3$, and $c_9 = -6k_0 k_2 + 3{k_1}^2/2 - 3k_1 {k_2}^2 + {k_2}^4$ by using coefficients of differential equation of the Weierstrass type elliptic curve ${u_x}^2 = 2u^3 + k_2 u^2 + k_1 u + k_0$.

For general higher order KdV equations, special 1-variable hyperelliptic solutions are known but elliptic solutions are not known so far. Since the same GL(2,$\mathbb{R}$)$\cong$ SO(2,1) Lie group structure and the same Bäcklund transformation exists even for higher order KdV equations, the existence of elliptic solutions will be guaranteed. We can show that elliptic solutions for all higher order KdV equations really exist by the following arguments: For the general $(2n + 1)$-th order KdV equation, it can be formulated in the Jacobi's inversion problem [19, 20], and it is known that there exist solutions expressed with the special 1-variable hyperelliptic theta function of the form $\vartheta(\pm 2x + d_1, d_2, \cdots, n)$ [20–24], which is shown to be the elliptic theta function according to the Mumford's argument [30]. We can say in another way. As the soliton solution $u(x,t) = u(X)$, $(X = \alpha x + \beta t_{2n+1} + \delta)$, which is expressed as the ratio of special 1-variable hyperelliptic theta functions, as it has the trivial periodicity $X \to X + 1$, $u(X)$ must be the trigonometric/hyperbolic or the elliptic function. Then it becomes the elliptic function according to the Mumford's argument. By using these facts, we can conclude that we always have the elliptic solutions for the general higher order KdV equations.

Further, without using the explicit form of the solution expressed with the $\wp$ function, we have shown that the KdV type Bäcklund transformation connects one solution to another solution of the same differential equation of the Weierstrass type elliptic curve.

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